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# Approximate Queueing System Model via N-dimensional Elementary Return Process

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あらまし

本論文では、計算機ネットワークなどの大規模な待ち合せシステムのモデル(近似)として、 $N$ 次元の基本復帰拡散過程を提案しその解析を行っている。大規模システム(ネットワークが主)の拡散近似は、Reiman 等[2]により、反射壁境界を用いた解析が行なわれており、極限重負荷定理( $\rho \rightarrow 1$ で近似が成立するという定理)が示されているが、 $\rho$ が小さければ、精度の面で問題がある。これを改良するため、本論文では Gelenbe [6] の結果を  $N$ 次元に拡張し、 $N$ 次元基本復帰過程の前進方程式を導く。応用として対称コンピュータ通信システムの系内容数の公식을導く。

## 1. Introduction

We consider an  $n$ -dimensional diffusion process  $Y_t$  that arises in conjunction with large queueing systems such as networks of queues, preemptive-resume queueing systems, etc.. Its state space consists of the interior and boundaries of the parallelotope of  $R^n$  (i.e.,  $0 \leq x_i \leq L_i$ ,  $i=1,2,\dots,n$ ). On the interior of this state space,  $Y_t$  behaves like an ordinary  $n$ -dimensional diffusion process (Brownian motion with drift). Whenever  $Y_t$  reaches one of the  $(n-1)$ -dimensional hyperplanes (for example,  $x_i=0$ ) it remains there for an exponentially distributed finite sojourn time and  $Y_t$  behaves this time as an  $(n-1)$ -dimensional diffusion process. After this finite sojourn time, a jump in the direction of  $x_i$ -axis occurs to a point whose  $x_i$ -coordinate is on the interior of  $(0, L_i)$  according to some probability density. The process then starts from scratch. The behavior of  $Y_t$  on the lower dimensional hyperplane is defined in the same way. The process thus defined is a natural extension of one-dimensional elementary return process [4].

Several works have been devoted to the diffusion approximation to large queueing systems (mainly, to open queueing networks). Kobayashi [10] wrote out the forward equation for open queueing networks with the rough boundary condition that the approximating process must be restricted to the nonnegative orthant. Harrison [7] and Harrison and Reiman [8] showed the precise boundary conditions for a tandem queue which the reflected Brownian motion should obey. It was also shown that their reflected Brownian motion satisfies the heavy traffic limit

theorem [2].

Although the heavy traffic limit theorem was proved for the reflected Brownian motion, it has the disadvantage of failing to model explicitly the probability of an empty queue [2]. With this motivation Gelenbe [6] proposed a heuristic approach for modeling empty queue probabilities by applying one-dimensional elementary return process. Our primary goal here is to extend his work to multi-dimensional case and specifically, to derive the stationary forward equations of the  $n$ -dimensional elementary return process as a model of large queueing systems.

In Section 2 and 3 we define the multi-dimensional elementary return process and derive the corresponding stationary forward equations. Using the result derived in Section 3, we then derive in Section 4 the partial differential equations for open networks of queues and the preemptive-resume queueing systems. Furthermore the important remark for discretization of the process is proposed. In Section 5, we discuss the mean of the queue length of the pairs of communicating computers by analyzing the partial differential equations derived in section 4.

A general notation shall be used through the paper that the infinitesimal volume elements of  $R^n$  is denoted  $dx = dx_1 dx_2 \dots dx_n$ .

## 2. Definition of the process

We shall define a multi-dimensional elementary return process  $Y_t$  or  $Y(t)$ ,  $t \geq 0$ . The most general case shall be discussed where the state space  $S^n$  of the process is the parallelotope of  $R^n$

$$0 \leq x_i \leq L_i, \quad L_i > 0 \quad (i=1,2,\dots,n).$$

Note that  $L_i$  may be infinite.  $S^n$  consists of its interior and the lower dimensional cells (boundaries). The number of the 0-cells (vertexes) of  $S^n$  is  $2^n$ . Generally the number of the  $(n-k)$ -cells of  $S^n$  is  $2^k \binom{n}{k}$ ,  $k=1,2,\dots,n$  and sum to  $3^n$ . We number these cells through so that the  $(n-k)$ -cells are denoted  $C_i$ ,  $(\sum_{j=1}^{k-1} 2^j \binom{n}{j}) + 1 \leq i \leq \sum_{j=1}^k 2^j \binom{n}{j}$  for  $k>0$  and  $C_0$  for  $k=0$ . Note that  $C_0$  represents the interior of  $S^n$ . We write  $C_j > (\geq) C_i$  when  $C_i$  is (not) a boundary of  $C_j$ . Let  $\pi^i$  be the natural projection map from  $R^n$  to  $R^{n-k}$  which is the sub-space of  $R^n$  parallel to the  $(n-k)$ -cell  $C_i$ . The index of the coordinate axis on  $\pi^j(R^n)$  which is orthogonal to  $\pi^i(R^n)$  is denoted  $i * j$  if  $C_j > C_i$ .

On the interior of  $S^n$ ,  $Y_t$  behaves as an  $n$ -dimensional Brownian motion which has the (constant coefficient) forward operator

$$L_0^* = \sum_{\phi, \phi'=1}^n \frac{1}{2} v_{\phi\phi'}^0 \frac{\partial^2}{\partial x_\phi \partial x_{\phi'}} - \sum_{\phi=1}^n m_\phi^0 \frac{\partial}{\partial x_\phi}$$

where the covariance matrix  $(v_{\phi\phi'}^0)$  is non-negative definite. Whenever  $Y_t$  reaches an  $(n-1)$ -cell  $C_i$ ,  $(1 \leq i \leq 2 \binom{n}{1})$  at time  $t_1$ , it stays there for a finite sojourn time  $T_i$  and behaves as an  $(n-1)$ -dimensional Brownian motion with the (constant coefficient) forward operator

$$L_i^* = \sum_{\phi, \phi' \neq i * 0} \frac{1}{2} v_{\phi\phi'}^i \frac{\partial^2}{\partial x_\phi \partial x_{\phi'}} - \sum_{\phi \neq i * 0} m_\phi^i \frac{\partial}{\partial x_\phi}$$

where the covariance matrix  $(v_{\phi\phi'}^i)$  is non-negative definite. If  $Y_t$  reaches on an  $(n-2)$ -cell  $C_j < C_i$  at  $t_1 + T_i$ , it stays there for an finite sojourn time  $T_j$  and behaves as an  $(n-2)$ -dimensional Brownian motion which has the (constant coefficient)

forward operator

$$L_j^* = \sum_{\phi \phi' \neq i \neq 0, j \neq i} \frac{1}{2} v_{\phi \phi'}^j \frac{\partial^2}{\partial x_{\phi}^2 \partial x_{\phi'}^2} - \sum_{\phi \phi' \neq i \neq 0, j \neq i} m_{\phi \phi'}^j \frac{\partial}{\partial x_{\phi}^2}$$

with the nonnegative definite covariance matrix  $(v_{\phi \phi'}^j)$ . The behavior of  $Y_t$  and the forward operator on the lower dimensional cells are defined in the same way except for the 0-cells (vertexes). If one of the 0-cells is reached then  $Y_t$  stays there for an exponentially distributed finite sojourn time. The finite sojourn time  $T_k$  on  $C_k$  ( $0 \leq k \leq 3^n - 1$ ) is exponentially distributed if any boundary cell  $C_i$  of  $C_k$  (i.e.,  $C_i < C_k$ ) is not reached. As soon as the exponential sojourn time lapses before hitting the boundaries, a jump in the direction of  $x_{k \star m}$ -axis occurs to the interior of  $C_m$  from  $C_k$ , ( $C_m > C_k$ ) according to some probability density function over  $0 < x_{k \star m} < L_{k \star m}$ .

Let  $\lambda_{km}$  be the positive constant which represents the rate at which jumps occur to a point on the interior of  $C_m$  from  $C_k$ , ( $C_m > C_k$ ). Then

$$\begin{aligned} & \text{Prob} \{ T_k > t \mid \text{any } C_i < C_k \text{ is not reached} \} \\ &= \exp \left( -t \sum_m \lambda_{km} \right) \end{aligned}$$

where the summation in the exponential function is taken over all  $m$  such that  $C_m > C_k$ . For our purpose, let the density function for the jump be the Dirac delta function  $\delta \{ x_{k \star m} - M(k, m) \}$  in which  $M(k, m)$  is defined as follows: let

$$(1) \quad \text{sgn}(k, m) = \begin{cases} +1 & \text{if } x_{k \star m} = 0 \text{ for } x \in C_k \\ -1 & \text{if } x_{k \star m} = L_{k \star m} \text{ for } x \in C_k \end{cases}$$

then

$$M(k, m) = \frac{1}{2} [1 + \text{sgn}(k, m)] + \frac{1}{2} [1 - \text{sgn}(k, m)] (L_{k \star m} - 1).$$

After the jump, the process starts from scratch. The process  $Y_t$  thus defined is the Markov process because there are no point with memory.

### 3. Stationary forward equation

Let the alternative representation of the process be

$$Y(t) = (k, \chi^k)_t$$

where  $k$  represents the index of the cell on which the process sojourns and  $\chi^k = \pi^k(Y)$ . We assume the process has the transition probability density defined by

$$p_t^{ij}(x^i, y^j) \Delta y^j$$

$$\cong \text{Prob} \{ (k, \chi^k)_t = (j, y^j), y^j \in \Delta y^j \mid (k, \chi^k)_0 = (i, x^i) \}$$

in which  $\Delta y^j$  represents the small volume element on  $C_j$ . We decide  $p_t^{ij}(x^i, y^j)$  represents the transition probability if  $C_j$  is a 0-cell.

Let  $x^k = \pi^k x$ . Let  $\mathcal{Y}_k$ ,  $k=0, \dots, 3^n-1$  be the set of continuous functions  $f_k: \pi^k(C_k) \rightarrow \mathbb{R}^1$  that are twice continuously differentiable in  $x^k$  except at  $z^k$  whose component  $x_i$  of  $\mathbb{R}^n$ , if contained, is 1 or  $L_i-1$  for some  $i$ . Let  $f_k(x^k) \in \mathcal{Y}_k$  be the density function on  $C_k$  that satisfies

$$\sum_{k=0}^{3^n-1} \int_{\pi^k C_k} f_k(x^k) dx^k = 1.$$

For the density function  $f_k$ , we define the transition operator by

$$(T_t^* f)_j(z^j) = \sum_k \int \pi^{k c_k} f_k(x^k) p_t^{kj}(x^k, z^j) dx^k$$

where  $f = (f_0, \dots, f_{3^n-1})$  and  $z^j \in \pi^j c_j$ . Then

the forward operator of the process is

$$(2) \quad A^* = \lim_{t \rightarrow 0^+} \frac{T_t^* - I}{t}$$

where  $I$  represents the unit operator on  $\mathcal{Y}_0 \times \mathcal{Y}_1 \times \dots \times \mathcal{Y}_{3^n-1}$ . The stationary forward equation of the process can be represented in the form

$$(3) \quad (A^* f)_i = 0, \quad i=0,1,\dots,3^n-1.$$

It will be convenient to define the differential operators

$$J_{\phi}^i [f_i(\cdot)] = -\frac{1}{2} \sum_{\phi'} v_{\phi\phi'}^i \frac{\partial f_i(x^i)}{\partial x_{\phi}} + m_{\phi}^i f_i(x^i)$$

and

$$H^{ik} [f_i(\cdot)] = \lim_{x_{k \star i} \rightarrow N(k,i)} J_{k \star i}^i [f_i(\cdot)]$$

where  $N(k,i) = (1/2)[1 - \text{sgn}(k,i)] L_{k \star i}$ ,  $i=1,\dots,3^n-1$ , and  $\phi, \phi'$  represent indexes of the components of  $x^i$  and the summation is taken over all such indexes. The following theorem gives the concrete form of the equation (3). We shall maintain all of the notation established earlier.

Theorem 1. Assume that  $Y_t$  has a stationary density function  $f_k(\pi^k x)$ ,  $k=0,1,\dots,3^n-1$ . Then it is the solution of the



equations

$$-L_0^* f_0(x) = \sum_{i=1}^{2n} \lambda_{i0} f_i(x^i) \delta(x_{i*0} - M(i,0))$$

.....

$$\begin{aligned} -L_k^* f_k(x^k) = & \sum_{m \text{ s.t. } c_m > c_k} \left\{ -\text{sgn}(k,m) H^{mk} [f_m(x^m)] \right. \\ & \left. - \lambda_{km} f_k(x^k) \right\} + \sum_{j \text{ s.t. } c_k > c_j} \lambda_{jk} f_j(x^j) \delta(x_{j*k} - M(j,k)) \\ & (1 \leq k \leq 3^n - 2^n - 1) \end{aligned}$$

.....

$$\begin{aligned} 0 = & \sum_{m \text{ s.t. } c_m > c_k} -\text{sgn}(k,m) H^{mk} [f_m(x^m)] - \lambda_{km} f_k \\ & (3^n - 2^n \leq k \leq 3^n - 1) \end{aligned}$$

with boundary conditions

$$\lim_{x_\phi \rightarrow 0} f_k(x^k) = \lim_{x_\phi \rightarrow L_\phi} f_k(x^k) = 0$$

where  $x_\phi$  is the component of  $x^k$ ,  $0 < k \leq 3^n - 1$ .

Remark. If all  $L_i$  are finite and there are no absorbing states,  $Y_t$  is positive recurrent and has a stationary density.

Proof. Let define

$$(a_t^* f)_i = (T_t^* f)_i - f_i(x^i)$$

for any  $f_i \in \mathcal{Y}_i$ ,  $(i=0, \dots, 3^n-1)$ . We need to show the concrete form of the equation

$$(4) \quad \lim_{t \rightarrow 0^+} \frac{1}{t} (a_t^* f)_i = 0.$$

Since the transition probability densities  $p_t^{kj}(x^k, z^j)$ , ( $k \neq j$ ,  $C_j$   $\neq C_k$ ) vanish with higher order than  $p_t^{ki}(x^k, z^i)$ , ( $C_i > C_k$ ) as  $t \rightarrow 0$ , we have

$$(5) \quad (a_t^* f)_i(z^i) = \sum_{k \text{ s.t. } C_i > C_k} \int \pi^k C_k f_k(x^k) p_t^{ki}(x^k, z^i) dx^k \\ + \left[ \int \pi^i C_i f_i(x^i) p_t^{ii}(x^i, z^i) dx^i - f_i(z^i) \right] \\ + \sum_{k \text{ s.t. } C_k > C_i} \int \pi^k C_k f_k(x^k) p_t^{ki}(x^k, z^i) dx^k, \\ + \xi(t), \quad (\text{for small } t > 0)$$

where  $\xi(t)/t \rightarrow 0$ , ( $t \rightarrow 0$ ). In the integrand of the first term of (5), the transition probability density is given by

$$p_t^{ki}(x^k, z^i) \\ = \int \pi^k C_k p_t^{kk}(x^k, z^k) \{1 - \exp(-\lambda_{ki} t)\} \delta\{\pi^k z^i - z^k, z_{k \star i} - M(k, i)\} dz^k \\ + O(t)$$

Hence the first term in the right hand side of (5) multiplied by  $1/t$  as  $t \rightarrow 0$  becomes

$$(6) \quad \sum_{k \text{ s.t. } C_i > C_k} \lambda_{ki} f_k(x^k) \delta(x_{k \star i} - M(k, i)).$$

Let  $q_t^i(x^i, z^i)$  be the transition probability density of the Brownian motion on  $C_i$  (with the forward operator  $L_i^*$ ). Then

$$p_t^{ii}(x^i, z^i) = \exp\left(- \sum_{k \text{ s.t. } C_k > C_i} \lambda_{ik} t\right) q_t^i(x^i, z^i).$$

Thus the second term in the right hand side of (5) can be written as follows:

$$(7) \quad \left[ \int \pi^{i c_i} f_i(x^i) q_t^i(x^i, z^i) dx^i - f_i(z^i) \right] \exp(- \sum_{j \text{ s.t. } c_j > c_i} \lambda_{ij} t) + f_i(z^i) \left[ \exp(- \sum_{j \text{ s.t. } c_j > c_i} \lambda_{ij} t) - 1 \right].$$

Letting  $t \rightarrow 0$  in (7) multiplied by  $1/t$ , we obtain

$$(8) \quad L_i^* f_i(x^i) = \sum_{j \text{ s.t. } c_j > c_i} \lambda_{ij} f_i(x^i).$$

Let

$$(9) \quad S_t = \sum_{i \text{ s.t. } c_k > c_i} \int \pi^{i c_i} \int \pi^{k c_k} f_k(x^k) p_t^{ki}(x^k, z^i) dx^k dz^i \\ = \sum_{k c_k} f_k(x^k) \left[ \sum_{i \text{ s.t. } c_k > c_i} \int \pi^{i c_i} p_t^{ki}(x^k, z^i) dz^i \right] dx^k,$$

the last equality resulting by interchanging orders of integration. Since the transition density functions  $p_t^{kj}(x^k, z^j)$ , ( $k=j$ ,  $c_j > c_k$ ) vanish with higher order than  $p_t^{ki}(x^k, z^i)$ , ( $c_k > c_i$ ) as  $t \rightarrow 0$ , we have

$$\sum_{i \text{ s.t. } c_k > c_i} \int \pi^{i c_i} p_t^{ki}(x^k, z^i) dz^i \\ = 1 - \int \pi^{k c_k} p_t^{kk}(x^k, z^k) dz^k - \sum_{j \text{ s.t. } c_j > c_k} \int \pi^{j c_j} p_t^{kj}(x^k, z^j) dz^j \\ + \mathcal{E}(t), \text{ (for small } t > 0),$$

where  $\lim_{t \rightarrow 0} \mathcal{E}(t)/t = 0$ . Thus

$$\begin{aligned}
 (10) \quad S_t = & \int \pi^{k c_k} f_k(z^k) dz^k - \int \int \pi^{k c_k} f_k(x^k) p_t^{kk}(x^k, z^k) dx^k dz^k \\
 & - \int \pi^{k c_k} \sum_{j \text{ s.t. } c_j > c_k} f_k(x^k) \int \pi^{j c_j} p_t^{kj}(x^k, z^j) dx^j dx^k \\
 & + \mathcal{E}(t), \quad (\text{for small } t),
 \end{aligned}$$

where  $\mathcal{E}(t)/t = 0$ , ( $t \rightarrow 0$ ). Using the same discussion as before, the first two terms and the last term in the right hand side of (10) multiplied by  $1/t$  as  $t \rightarrow 0$  become

$$\int \pi^{k c_k} \left[ -L_k^* f_k(x^k) + \sum_{j \text{ s.t. } c_j > c_k} \lambda_{kj} f_k(x^k) \right] dx^k$$

and

$$- \int \pi^{k c_k} \sum_{j \text{ s.t. } c_j > c_k} \lambda_{kj} f_k(x^k) dx^k, \quad (t \rightarrow 0)$$

respectively. Thus

$$\lim_{t \rightarrow 0} S_t/t = \int \pi^{k c_k} -L_k^* f_k(x^k) dx^k$$

or in the alternative expression,

$$(11) \quad = \int \pi^{k c_k} \operatorname{div} J^k[f_k(\cdot)] dx^k.$$

where  $\operatorname{div} = (\partial/\partial x^{\phi})$ . Applying the well known divergence theorem to (11), we obtain

$$(12) \quad \lim_{t \rightarrow 0} S_t/t = \sum_{i \text{ s.t. } c_k > c_i} \int \pi^i c_i^{-\text{sgn}(i,k)} H^{ki}[f_k] dz^i.$$

On the other hand from the first equation of (9) multiplied by  $1/t$  as  $t \rightarrow 0$ , we have

$$(13) \quad \lim_{t \rightarrow 0} S_t/t = \sum_{i \text{ s.t. } c_k > c_i} \int \pi^i c_i \lim_{t \rightarrow 0} \frac{1}{t} \int \pi^k c_k f_k(x^k) p_t^{ki}(x^k, z^i) dx^k dz^i$$

Comparing (12) with (13), we have

$$(14) \quad -\text{sgn}(i,k) H^{ki}[f_k] = \lim_{t \rightarrow 0} (1/t) \int \pi^k c_k f_k(x^k) p_t^{ki}(x^k, z^i) dx^k,$$

for  $c_k > c_i$ . From (4), (6), (8), and (14) we obtain the forward equations in the theorem. Note that since the  $n$ -cell and the  $0$ -cells have no upper cells and lower cells respectively, we can obtain the equations putting the related functions to be  $0$ . Finally, since the boundaries act as absorbing boundaries for the diffusion process on  $c_k$ , ( $k=0, \dots, 3^{n-1}$ ) until the next jump occurs, we obtain the boundary condition in the theorem. This completes the proof.

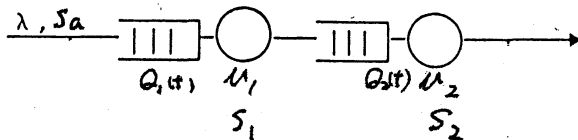


FIG. 1.

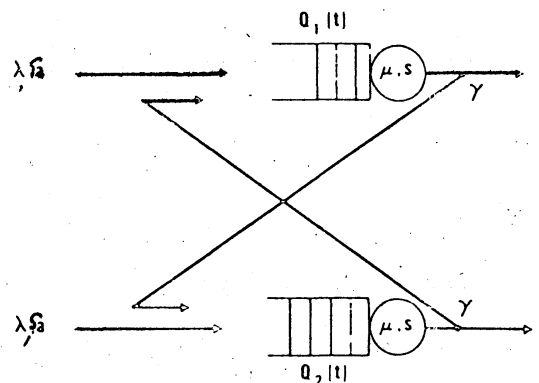


Fig. 2

#### 4. Approximation for some queueing processes

We consider in this section some examples of queues which can be modeled with the two-dimensional elementary return process. Although we restrict our attention to the two-dimensional case for simplicity, most of techniques we use will be extended to higher dimensions.

From Theorem 1, the partial differential equations of the two-dimensional elementary return process on  $R_+^2$  is written as follows;

$$\operatorname{div} J^0(x) = \sum_{i=1}^2 \lambda_{i0} f_i(x_{3-i}) \delta(x_i - 1)$$

$$(15) \quad (\partial / \partial x_{3-i}) J^i(x_{3-i}) = \lim_{x_i \rightarrow 0} -J_i^0(x) - \lambda_{i0} f_i(x_{3-i}) + \lambda_{3i} f_3 \delta(x_{3-i} - 1), \quad (i=1,2)$$

$$0 = \lim_{x_1 \rightarrow 0} -J^2(x_1) + \lim_{x_2 \rightarrow 0} -J^1(x_2) - (\lambda_{31} + \lambda_{32}) f_3,$$

$$\lim_{x_i \rightarrow 0} f_0(x_1, x_2) = 0, \quad (i=1,2)$$

$$\lim_{x_i \rightarrow 0} f_i(x_{3-i}) = 0, \quad (i=1,2)$$

where  $f_3$  is the probability mass at the origin,

$$J_i^0(x) = m_i^0 f_0 - \frac{1}{2} \sum_{j=1}^2 v_{ij}^0 \frac{\partial f_0}{\partial x_j}$$

$$J^0 = (J_i^0)$$

and

$$J^i(x_{3-i}) = m^i f_i - \frac{1}{2} v^i \frac{\partial f_i}{\partial x_{3-i}}, \quad (i=1,2).$$

We determine the coefficients of (15) based on the fact that the diffusion process approximates the number in queue as soon as the number in queue moves from zero to one.

Example 1. First we consider the tandem queues of Figure 1 which is composed of two single server stations arranged in tandem. Customers arrive at station 1 according to a renewal process whose interarrival times have mean  $1/\lambda$  and variance  $S_a$ . Having completed service there, they proceed to station 2, and after completing service at station 2 they leave the system. Each station has a single server, and service times at station  $i$ , ( $i=1,2$ ) are i.i.d. with mean  $1/\mu_i$  and variance  $S_i$ . Let  $Q_i(t)$  denote the number of customers in station  $i$  at time  $t$ , and let  $Q(t)=(Q_1(t), Q_2(t))$ . The coefficients of (15) are determined as follows.

(i) As soon as a busy period ( $Q_1(t)>0$ ,  $Q_2(t)>0$ ) starts, the process  $Q(t)$  approximately behaves as the ordinary two-dimensional diffusion process with the coefficients determined by "central-limit-theorem"-type argument [10]. Thus from [10]

$$m^0 = \begin{pmatrix} \lambda - \mu_1 \\ \mu_1 - \mu_2 \end{pmatrix}$$

$$v^0 = \begin{pmatrix} \lambda^3 S_a + \mu_1^3 S_1 & -\mu_1^3 S_1 \\ -\mu_1^3 S_1 & \mu_1^3 S_1 + \mu_2^3 S_2 \end{pmatrix}.$$

(ii) On the half-line  $Q_1=0(>0)$ ,  $Q_2>0(=0)$ ,  $Q_2(t)$  ( $Q_1(t)$ ) behaves approximately as the one-dimensional diffusion process with infinitesimal mean  $m^1=-\mu_2$  ( $m^2=\lambda-\mu_1$ ) and infinitesimal variance  $v^1=\mu_2^3 S_2$  ( $v^2=\lambda^3 S_a + \mu_1^3 S_1$ ). The sojourn time on this half-line

behaves approximately as the Markovian arrival with the rate  $\lambda$  ( $\mu_1$ ). Thus we set  $\lambda_{10} = \lambda$  and  $\lambda_{20} = \mu_1$ .

(iii) At the origin  $Q_1=0$ ,  $Q_2=0$ ,  $Q_1(t)$  approximately behaves as the Markovian arrival with rate  $\lambda$ . Thus we can set  $\lambda_{32} = \lambda$ . Since there is no arrival at station 2 when  $Q_1=0$ , we set  $\lambda_{31} = 0$ .

Example 2. The second example is the pairs of identical computers in reciprocal communication (see Figure 2). This model was discussed by Foshini [5] under the heavy traffic condition. He solved the mean queue length applying the two-dimensional reflected diffusion process [8]. We will also discuss the mean queue length of this model in the next section.

The mean rate and the variance of exogenous interarrivals of jobs are denoted by  $\lambda$  and  $S_a$  respectively. Service times at each station are i.i.d. with mean  $1/\mu$  and variance  $S$ . Having completed service at each station, jobs proceed to other computer for services with probability  $\gamma$  and with probability  $1-\gamma$  jobs leaves the system. Using the same discussion in Example 1, the parameters of this model become as follows;

$$m_1^0 = m_2^0 = \lambda - \mu(1-\gamma)$$

$$v_{11}^0 = v_{22}^0 = \lambda^3 S_a + \mu^3 S + \mu\gamma(1-\gamma + \gamma\mu^2 S)$$

$$v_{12}^0 = v_{21}^0 = -2\mu^3 S\gamma$$

$$m^1 = m^2 = \lambda - \mu$$

$$v^1 = v^2 = \lambda^3 S_a + \mu^3 S$$

$$\lambda_{10} = \lambda_{20} = \lambda + \gamma\mu$$

and

$$\lambda_{31} = \lambda_{32} = \lambda$$



We have discussed thus far only the queueing network models. The multi-dimensional elementary return process is applicable to another kind of queueing systems as described in the next example.

Example 3. Consider the preemptive-resume queueing systems which handle two types of customers. In this system, one type of customers is given a preemptive priority over the other type and served by a single server. Each customers arrive at the queue according to a renewal process whose interarrival times have mean  $1/\lambda_i$  and variance  $a_i$ ,  $i=1$  for high priority class and  $i=2$  for low priority class. The service times for class  $i$  are i.i.d. with mean  $1/\gamma_i$  and variance  $S_i$ . The same discussion as before yields

$$m^0 = \begin{pmatrix} \lambda_1 - \gamma_1 \\ \lambda_2 \end{pmatrix}$$

$$v^0 = \begin{pmatrix} \lambda_1^3 a_1 + \gamma_1^3 s_1 & 0 \\ 0 & \lambda_2^3 a_2 \end{pmatrix}$$

$$m^1 = \lambda_1 - \gamma_1$$

$$v^1 = \lambda_1^3 a_1 + \gamma_1^3 s_1$$

$$m^2 = \lambda_2 - \gamma_2$$

$$v^2 = \lambda_2^3 a_2 + \gamma_2^3 s_2$$

$$\lambda_{32} = \lambda_{10} = \lambda_2$$

and

$$\lambda_{31} = \lambda_{20} = \lambda_1.$$

Notice that since the changes in the number of the class 1 customers is completely independent of the class 2 customers, the covariance between these two classes is zero.

Finally we remark the discretization of the density. A direct method (see [6]) is given by setting

$$P(0,0) = f_3$$

$$P(i,0) = \int_{i-1}^i f_1(x) dx, \quad i > 0$$

$$P(0,j) = \int_{j-1}^j f_2(x) dx, \quad j > 0$$

and

$$P(i,j) = \int_{i-1}^i \int_{j-1}^j f_0(x,y) dx dy, \quad i,j > 0$$

where  $P(i,j)$  is the approximate statistical-equilibrium probability that  $Q_1=i$  and  $Q_2=j$  customers simultaneously exist in the queue. This discretization, however has the disadvantage of failing to model the probability that one customer exists in the queue. This disadvantage is caused by the jump that reduces the occupation time of the section between  $x_i=0$  and 1,  $i=1,2$ .

The alternative method of discretization which overcomes this disadvantage is given by

$$P(0,0) = f_3$$

$$P(i,0) = K_1 f_1(i), \quad i > 0$$

$$P(0,j) = K_2 f_2(j), \quad j > 0$$

and

$$P(i,j) = K_0 f_0(i,j), \quad i,j > 0$$

where

$$K_1 = \int_0^\infty f_1(x) dx / \sum_{i=1}^\infty f_1(i)$$

$$K_2 = \int_0^\infty f_2(x) dx / \sum_{j=1}^\infty f_2(j)$$

and

$$K_0 = \int_0^\infty \int_0^\infty f_0(x, y) dx dy / \sum_{i,j=1}^\infty f_0(i, j) .$$

This method is based on the concept of local occupation time. The jump does not have the effect any more to reduce the probability of one customer in each queue.

#### 5 Formulas for mean

In this section we discuss the formulas for the equilibrium mean of the symmetric pairs of communicating computers. From Example 2 of section 4, the partial differential equations with the boundary conditions are rewritten as follows:

$$(16) \quad \frac{1}{2} A \frac{\partial^2 f_0}{\partial x_1^2} + B \frac{\partial^2 f_0}{\partial x_1 \partial x_2} + \frac{1}{2} A \frac{\partial^2 f_0}{\partial x_2^2} - c \frac{\partial f_0}{\partial x_1} - c \frac{\partial f_0}{\partial x_2} \\ = -\beta f_1(x_1) \delta(x_2 - 1) - \beta f_2(x_2) \delta(x_1 - 1)$$

$$(17) \quad -\frac{1}{2} D \frac{\partial^2 f_i}{\partial x_i^2} + E \frac{\partial f_i}{\partial x_i} = -H_{3-i}(x_i) - \gamma f_i(x_i) + \lambda f_3 \delta(x_i - 1) \\ (i=1, 2)$$

$$(18) \quad K_1 + K_2 + 2 \lambda f_3 = 0$$

$$f_0(x_1, 0) = f_0(0, x_2) = 0$$

$$f_1(0) = f_2(0) = 0$$

where

$$H_{3-i}(x_i) = \lim_{x_{3-i} \rightarrow 0} -\frac{1}{2} B \frac{\partial f_0}{\partial x_i} - \frac{1}{2} A \frac{\partial f_0}{\partial x_{3-i}} + c f_0$$

$$K_i = \lim_{x_i \rightarrow 0} -\frac{1}{2} D \frac{\partial f_i}{\partial x_i} + E f_i, \quad (i=1, 2)$$

and where

$$A = \lambda^3 s_a + \mu^3 s + \mu \gamma (1 - \gamma + \gamma \mu^2 s)$$

$$B = -2 \mu^3 s \gamma$$

$$C = \lambda - \mu (1 - \gamma)$$

$$D = \lambda^3 s_a + \mu^3 s$$

$$E = \lambda - \mu$$

$$\beta = \lambda + \gamma \mu$$

and  $\lambda, s_a, \mu, s, \gamma$  which are established in Example 2 are maintained.

Let define the Laplace transformation

$$L_0(s_1, s_2) = \int_0^\infty \int_0^\infty f_0(x_1, x_2) \exp(-s_1 x_1 - s_2 x_2) dx_1 dx_2$$

$$L_i(s_{3-i}) = \int_0^\infty f_i(x_{3-i}) \exp(-s_{3-i} x_{3-i}) dx_{3-i} \quad (i=1,2)$$

and

$$F_i(s_{3-i}) = \int_0^\infty H_i(x_{3-i}) \exp(-s_{3-i} x_{3-i}) dx_{3-i} \quad (i=1,2)$$

Taking the two-dimensional Laplace transformation of (16), we have

$$\begin{aligned} (19) \quad & \left[ \frac{1}{2} A s_1^2 + B s_1 s_2 + \frac{1}{2} A s_2^2 - C s_1 - C s_2 \right] L_0(s_1, s_2) \\ & + F_1(s_2) + F_2(s_1) \\ & = -\beta L_1(s_2) \exp(-s_1) - \beta L_2(s_1) \exp(-s_2) . \end{aligned}$$

Furthermore Laplace transformation of (17) and use of (18) yield

$$\begin{aligned} (20) \quad & F_i(s_{3-i}) = \left[ \frac{1}{2} B s_{3-i}^2 - C s_{3-i} - \beta \right] L_i(s_{3-i}) \\ & - \lambda f_3(1 - \exp(-s_{3-i})) . \end{aligned}$$

Let the power series expansion for  $L_0, L_i, i=1,2$  be

$$L_0(s_1, s_2) = a_0 + a_1 s_1 + a_1 s_2 + a_2 s_1 s_2 + \dots$$

and

$$L_2(s) = L_1(s) = b_0 + b_1 s + \dots$$

Substitute (20) into (19) and the power series expansion for  $L_0$ ,  $L_1$ ,  $L_2$  into the equation thus derived to get the coefficients of the  $s_1^2$ ,  $s_1 s_2$ , and  $s_1$ . Then we obtain

$$\frac{1}{2} A a_0 - C a_1 + \frac{1}{2} D b_0 - E b_1 + \lambda f_3/2 = -\frac{\beta}{2} b_0 \quad (21)$$

$$B a_0 - 2 C a_1 = 2 \beta b_1$$

and

$$-C a_0 - E b_0 - \lambda f_3 = \beta b_0 \quad (22)$$

Notice that

$$a_0 = L_0(0,0) = \int_0^\infty \int_0^\infty f_0(x_1, x_2) dx_1 dx_2$$

$$b_0 = L_1(0) = L_2(0) = \int_0^\infty f_1(x_2) dx_2$$

and thus

$$a_0 + 2b_0 + f_3 = 1 \quad (23)$$

Let the stationary probability that the process sojourns on  $x_i=0, i=1,2$  be

$$1 - \rho = b_0 + f_3 \quad (24)$$

Then we have from (22), (23), and (24)

$$\lambda (1 - \rho) + C \rho = b_0 (C + \lambda - E - \beta) \quad (25)$$

The right hand side of (25) vanishes and we obtain

$$\rho = \lambda / \mu (1 - \gamma) \quad (26)$$

This coincides the traffic intensity at each queue [2]. Although our approach is heuristic, we have had the precise modeling of

empty queue probability as well as the one-dimensional case [6].

Notice that the mean  $E[x_1] = E[x_2]$  is given by

$$\begin{aligned} E[x_1] &= \int_0^\infty x_1 \left[ \int_0^\infty f_0(x_1, x_2) dx_2 + f_2(x_1) \right] dx_1 \\ &= - \frac{\partial L_0}{\partial s_1}(0,0) - \frac{\partial L_2}{\partial s_1}(0) = -a_1 - b_1. \end{aligned}$$

We have thus proved the following result.

Theorem 2. Suppose that the equations (16), (17), and (18) have a stationary solution. Then the mean  $E[x_i]$  satisfies

$$(27) \quad E[x_i] = \frac{-1}{2\beta C - 2CE} [(\beta - C) \{ A(2\rho + m - 1) + (D + \beta)(1 - \rho - m) + \lambda m \} - (E\beta - BC)(2\rho + m - 1)]$$

where  $m$  represents the probability mass at the origin.

Theorem 2 states that if the steady state probability of empty queues is known, then the approximate mean queue length can be given by (27). Since the approximation is made assuming the Markovian arrival at the empty queues, it is reasonable selection to set

$$(28) \quad m = (1 - \rho)^2,$$

the equality being due to Jackson network result.

Assuming (28), we have

$$\begin{aligned} (29) \quad E[x_i] &= [ \gamma \rho^2 (1 - \gamma - \gamma \mu^2 s) + (\rho + \gamma)(1 - \rho)\rho \\ &\quad + \rho(\lambda^2 \rho s_a + \mu^2 s) + \rho(1 - \rho)^2 \\ &\quad - \rho^2 \{ (\rho - 1)(\rho + \gamma) + 2\mu s \gamma (\lambda - \mu(1 - \gamma)) \} ] / \\ &\quad 2(\gamma + 1)(1 - \rho - \gamma) \end{aligned}$$

For uncoupled M/G/1 system, from the Pollaczek-Khinchin formula the mean queue length is

$$\rho \left\{ 1 + \frac{\rho(1 + \mu^2 s)}{2(1 - \rho)} \right\}$$

while (29) becomes

$$E[x_i] = \rho \left\{ \frac{1 + \rho^2}{2} + \frac{\rho + \mu^2 s}{2(1 - \rho)} \right\}.$$

This appears to be a better approximation than

$$\frac{\rho + \mu^2 s}{2(1 - \rho)}$$

which is the result of Foschini [5] derived by means of the two-dimensional reflected diffusion process approximation [7].

## 6 Concluding remarks

Computing the density  $f$  of our basic differential equation (15) appears to be difficult problem. The most simple form solution can be derived as a product form. It will be easy to show that if (and only if) the covariance coefficients  $v_{ij}^0$ ,  $i \neq j$  are zero and  $v_{ii}^0 = v^{3-i}$ ,  $m_i^0 = m^{3-i}$ , ( $i=1,2$ ),  $\lambda_2 = \lambda_{10} = \lambda_{32}$ ,  $\lambda_1 = \lambda_{20} = \lambda_{31}$ , then there exists the separable exponential form solution:

$$f_{3-i}(x_i) = f_3 \exp(G_{3-i} x_i) \frac{2\lambda_{3-i}}{v^{3-i}} \int_0^{x_i} \exp(-G_{3-i} x) F(x) dx$$

and

$$f_0(x) = \frac{4\lambda_1 \lambda_2}{v^1 v^2} f_3 \exp(G_1 x_1 + G_2 x_2) \prod_{i=1}^2 \int_0^{x_i} \exp(-G_i x) F(x) dx$$

where  $G_i = 2m^i/v^i$  and

$$F(x) = \begin{cases} -1, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

This solution, however, does not give any additional useful

information on the stationary distribution. Although it is difficult to find out even the special solution of (15), it seems worth-while to attach this problem for the purpose of its practical utilization.

#### References

1. Bronstein, N and Semendiaev, K. A., A Guide-Book to Mathematics for Technologists and Engineers, Pergamon Press, Ltd., Oxford, 1963.
2. Coffman E.G. and Reiman M.I., "Diffusion Approximations for Computer/Communication Systems" International Workshop on Appl. Math. and Performance Reliability Models of Computer Communication Systems, Univ. of Pisa. (1983).
3. Cox, D.R. and Miller, H.D., The Theory of Stochastic Processes, Chapman and Hall Ltd., New York, 1965.
4. Feller, W., "Diffusion processes in one dimension." Trans. Amer. Math. Soc. 77(1954), 1-31.
5. Foschini, G. J., "Equilibria for Diffusion Models of Pairs of Communicating Computers-Symmetric Case," IEEE Trans. Infor. Theory, IT-28, 273-284.
6. Gelenbe, E., "On Approximate Computer System Models," J. Assoc. Comput. Mach., Vol22(1975), pp. 261-269.
7. Harrison, J.M., "The Heavy Traffic approximation for Single Server Queues in Series," J. Appl. Prob. Vol.10(1973), pp.613-629.



8. Harrison, J.M. and Reiman, M.I. "On the distribution of multi-dimensional reflected Brownian motion" SIAM J. Appl. Math. Vol. 41, No.2, (1981), pp345-361.
9. Kleinrock, L., Queueing Systems, Vol. II: Computer Applications, Wiley, New York, 1976.
10. Kobayashi, H., "Application of the Diffusion Approximation to Queueing Networks I: Equilibrium Queue Distributions," J. Assoc. Compute. Mach., Vol.21(1974), pp.316-328.

## 謝 辞

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